# Econ 2261 – Problem Set III

Provide a full justification for all your answers.
Due via OWL on 4/3 before 23:50 EST (UTC-5).

**1.** The government must decide whether to *build* a bridge or *not*. In order to build the bridge, the government must spend 10 million dollars. There are $n > 2$ individuals. Individual $i$'s private benefit from building the bridge is $v_i$.

(a) When is it efficient to build the bridge? Then the sum of individual values is greater than 10 million dollars.

(b) Find the VCG transfers for this problem. Say that individual $j$ is *pivotal* when $\sum_i v_i > 10$ and $\sum_{i \neq j} v_i < 10$. In other words, individual $j$ is pivotal if: ($i$) it is efficient to build the bridge, and ($ii$) it would not be efficient to build the bridge if $j$ was not a member of society. An individual only has a non-zero transfer if they are pivotal. The transfer for pivotal individuals is $t_j^{\text{VCG}} = 10 - \sum_{i \neq j} v_i$.

(c) Is there an efficient mechanism for this problem which *never* runs a deficit? *No*. Suppose that all individuals value is greater than 10. Then, it is efficient to build the bridge, but nobody is pivotal. In that case, the VCG mechanism runs a deficit. In fact, for this problem, the VCG mechanism builds a deficit whenever the bridge is built (why?). The VCG mechanism is the efficient mechanism that runs the smallest deficit.

**2.** Anna and Bob have two cars, a red car and a blue car. They need to decide who gets the red car, and who gets the blue car. Their values for each of the two cars are summarized in Table 1.

(a) What is the efficient outcome? The available alternatives are (1) to give Anna the red car and Bob the blue car, or (2) to give Ana the blue car and Bob the red car. The first alternative would result in a total value $v_A(\text{Red}) + v_B(\text{Blue})$. The second alternative would result in a total value $v_A(\text{Blue}) + v_B(\text{Red})$. Since in this class we maintain the assumption of quasilinear utilities, an outcome is efficient if and only if it is utilitarian. Hence, the efficient outcome is alternative (1) if $v_A(\text{Red}) + v_B(\text{Blue}) > v_A(\text{Blue}) + v_B(\text{Red})$, and alternative (2) otherwise.

(b) Find the VCG transfers for this problem. Suppose that alternative (1) is chosen. Then, the VCG transfers would be

$$t_A^{\text{VCG}} = \max\{v_B(\text{Red}), v_B(\text{Blue})\} - v_B(\text{Blue}),$$

and

$$t_B^{\text{VCG}} = \max\left\{v_A(\text{Red}), v_A(\text{Blue})\right\} - v_A(\text{Red}).$$

(c) Is there an efficient mechanism for this problem which *never* runs a deficit? *Yes.* Anna's and Bob's VCG transfers are always positive. Hence, the VCG mechanism never runs a deficit.

| | Red | Blue |
|---|---|---|
| Anna | $v_A(\text{Red})$ | $v_A(\text{Blue})$ |
| Bob | $v_B(\text{Red})$ | $v_B(\text{Blue})$ |

**Table 1** – Preferences over cars

**3.** Suppose that the drilling rights for a specific location are being auctioned. There are two bidders. Suppose that the value of the oil field is either high ($v = 100$) or low ($v = 0$), with each of these values being equally likely. Moreover, suppose that each bidder observes a signal $x_i$ that could be promising or discouraging. Signals are noisy. Conditional on the value of the field being high, the probability of an optimistic signal is 3/4. Conditional on the value of the field being low, the probability of an optimistic signal is 1/4.

(a) What is the expected value of the field for a bidder who observes a high signal? [*hint*: use Bayes rule] Let $x_i = 1$ denote an optimistic signal, and $x_i = 0$ denote a pessimistic signal. Using Bayes rule yields

$$\begin{aligned}\Pr(v = 100|x_i = 1) &= \frac{\Pr(v = 100)\Pr(x_i = 1|v = 100)}{\Pr(v = 100)\Pr(x_i = 1|v = 100) + \Pr(v = 0)\Pr(x_i = 1|v = 0)} \\ &= \frac{1/2 \cdot 3/4}{1/2 \cdot 3/4 + 1/2 \cdot 1/4} = \frac{3}{4}.\end{aligned}$$

The expected value of the field after observing an optimistic signal is

$$\mathbb{E}[\,v|x_i = 1\,] = \Pr(v = 100|x_i = 1) \cdot 100 + \Pr(v = 0|x_i = 1) \cdot 0 = \frac{3}{4} \cdot 100 = 75$$

(b) What is the expected value of the field for a bidder who observes a high signal, and realizes that the other bidder received a pessimistic signal? Let

$E$ be the event that bidder $i$ observed an optimistic signal, and bidder $-i$ observed a pessimistic signal. Note that

$$\Pr(E|v=100) = \Pr(x_i = 1|v=100)\Pr(x_{-i}=0|v=100) = \frac{3}{4}\frac{1}{4} = \frac{3}{16}.$$

Similarly, $\Pr(E|v=0) = 3/16$. Hence, it follows from Bayes rule that

$$\begin{aligned}\Pr(v=100|E) &= \frac{\Pr(v=100)\Pr(E|v=100)}{\Pr(v=100)\Pr(E|v=100)+\Pr(v=0)\Pr(E|v=0)} \\ &= \frac{1/2\cdot 3/16}{1/2\cdot 3/16 + 1/2\cdot 3/16} = \frac{1}{2}.\end{aligned}$$

The expected value of the field conditional on $E$ is

$$\mathbb{E}[\,v|E\,] = \Pr(v=100|E)\cdot 100 + \Pr(v=0|E)\cdot 0 = \frac{1}{2}\cdot 100 = 50$$

(c) Suppose that the drilling rights are auctioned via a sealed-bid second-price auction, and the two bidders are rational. Would the bidders bid a number greater than, less than, or equal to their expected value for the field? *(Weakly) less than.* Bidders with optimistic signals will big greater amounts than bidders with pessimistic signals, because they think the field is more profitable. Hence, a bidder is more likely to win the auction if the other bider observed a pessimistic signal. Therefore, the expected value of the object conditional on winning the auction is lower than the expected value at the moment of bidding. Rational agents will anticipate that and bid a smaller amount than their expectation of the value.

**4.** Suppose an object is to be allocated to either Bob, Charlie, or David. Whoever receives the object, it will generate consumption externalities for the other individuals in accordance with Table 2.

(a) What is the efficient outcome? The total social value from each alternative is given by: $\tilde{v}_B = v_B - 6$, $\tilde{v}_C = v_C - 4$, and $\tilde{v}_D = v_D - 6$, respectively. The efficient outcome is to choose the alternative with the highest social value.

(b) Find the VCG transfers for this problem.

(c) Is there an efficient mechanism for this problem which *never* runs a deficit?

**5.** Consider the roommate problem from the lecture notes. For each of the following mechanisms, indicate whether it is incentive compatible for Gary, incentive

| | a=B | a=C | a=D |
|---|---|---|---|
| $v_B(a)$ | $v_B$ | $-2$ | $-1$ |
| $v_C(a)$ | $-5$ | $v_C$ | $-5$ |
| $v_D(a)$ | $-1$ | $-2$ | $v_D$ |

**Table 2** – Negative consumption externality

compatible for Frank, budget balanced and/or Pareto efficient.

(a) First, Gary reports how much he is willing to pay ($\hat{v}_G$). Then, after hearing Bob's report, Frank announces how much he is willing to pay ($\hat{v}_F$). They buy the machine if and only if $\hat{v}_F + \hat{v}_G \geq 1000$. If they but the machine, Frank pays $1000 - \hat{v}_G$, and Gary pays $\hat{v}_G$.

IC *No.* Suppose that Gary thinks that $v_F = 500$ and that Frank will report truthfully. Also suppose that $v_G = 900$. If Gary reports truthfully he would have to pay 900. If he reports $\hat{v}_F = 500$, he would only have to pay 500.

BB *Yes.* Whenever they buy the machine, the transfers add up to 1000.

PE *Yes.* The mechanism chooses the efficient outcome given the *reported* values. (However, since the mechanism is not IC, the outcome might not be efficient given the true values.)

(b) Frank and Gary simultaneously report their values. Each pays 500, regardless of the reports, and regardless of whether they buy the machine. They buy the machine if and only if $\hat{v}_F + \hat{v}_G \geq 1000$.

IC *No.* Since the roommates will pay whether they buy the machine or not, it is weakly dominant for them to always report that their value is greater than 1000.

BB *Yes.* The mechanism never runs a deficit.

PE *Yes.* The mechanism chooses the efficient outcome given the *reported* values. (However, since the mechanism is not IC, the outcome might not be efficient given the true values.)

(c) They buy the machine and Frank pays 1000, if Frank's value is greater or equal than 1000. Otherwise, they do not buy the machine and nobody pays anything.

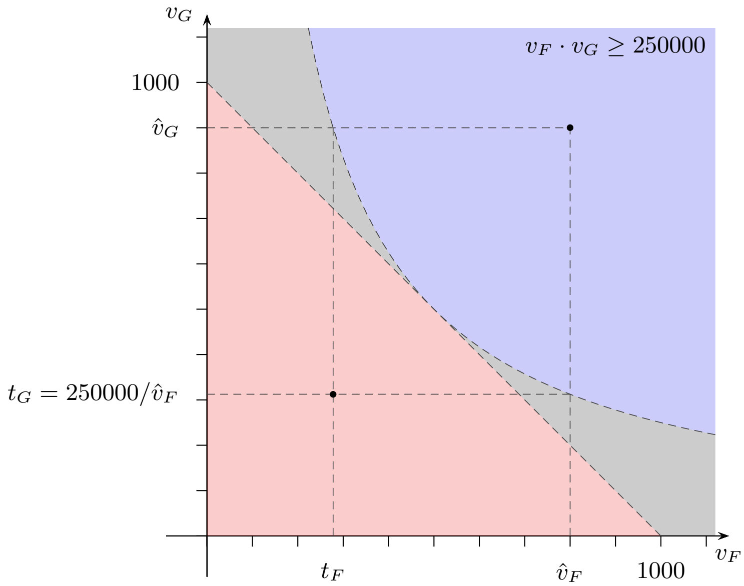


**Figure 1** – Mechanism for 5.(c)

- IC *Yes.* Only Frank's report affects the outcome. If the machine is bought Frank will have to pay 1000. Frank is willing to do so precisely when his value is greater or equal than 1000.
- BB *Yes.* When they buy the machine, Frank covers the entire amount.
- PE *No.* If $v_F + v_G > 1000$ but $v_F < 1000$, the outcome of the mechanism is inefficient.

(d) Frank and Gary simultaneously announce their values. They buy the machine if $\hat{v}_F \cdot \hat{v}_G \geq 250000$. If they buy the machine, Gary pays $250,000/\hat{v}_F$ and Frank pays $250,000/\hat{v}_G$.

- IC *Yes.* The zone where they buy the machine extends northeast, and each person pays the lowest value that would lead to buying the machine given their roommate's report. See figure 1.
- BB No. If $\hat{v}_G = 800$ and $\hat{v}_G = 900$, then $t_F + t_G = 590.2\bar{7} < 1000$. See Figure 1.
- PE No. See Figure 1.

(e) Frank and Gary simultaneously announce their values. They buy the machine if $(\hat{v}_F - 800)^2 + (\hat{v}_G - 800)^2 \leq 40000$. If they buy the machine, each pays

$$t_i = 800 - \sqrt{40000 - (\hat{v}_i - 800)^2}$$

[*hint*: draw a picture of this mechanism]

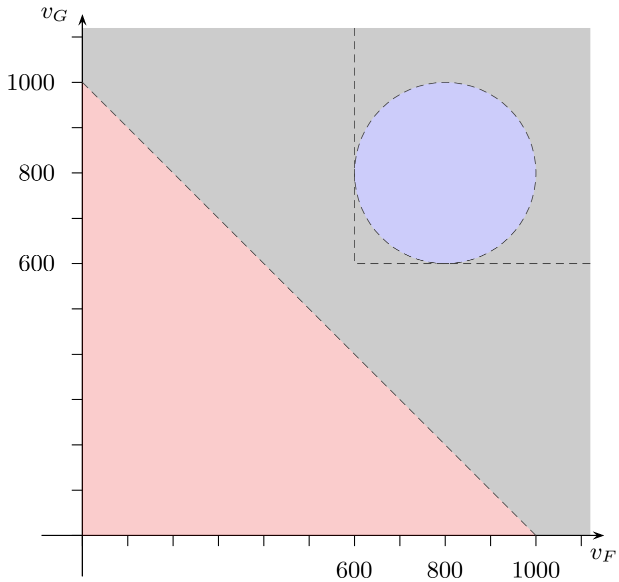


**Figure 2** – Mechanism for 5.(d)

IC *No.* The trade region does not extend northeast. See Figure 2.

BB *Yes.* The mechanism uses the transfer rule we discussed in class for IC mechanisms, and the trade region is contained in a rectangle above the budget line. See Figure 2.

PE *No.* See Figure 2.