Econ 9602 – Problem Set I

Due on 01/22

1. There are two restaurants named A and B. There prior probability that A is better than B is $q \in (1/2, 1)$, and the prior probability that B is better than A is $(1-q)$. There are countably many customers indexed by $i = 1, 2, \ldots$. Each customer observes one private signal taking one of two possible values α and β. The probability of observing α when A is the better restaurant is $\Pr(\alpha|A) = p > q$. Likewise, the probability of observing β when B is the better restaurant is $\Pr(\beta|B) = p$. The signals of different customers are independent conditional on the state. Customers update their beliefs using Bayes' Rule, and they go to the restaurant which is better with higher (posterior) probability.

(a) Which restaurant would a customer choose after observing signal α and no other information?

(b) Suppose that customer 1 observes α and customer 2 observes β. What is the posterior probability that A is the better restaurant conditional on these signals?

(c) Suppose that customer 1 observes α and customers 2 and 3 observe β. What is the posterior probability that A is the better restaurant conditional on these signals?

For the rest of the problem suppose that customers choose which restaurant to go to sequentially, and the choice of each customer is publicly observed by all other customers.

(d) If customer 1 observes α and every other customer observe signal β, what will be the outcome?

(e) What is the ex-ante probability that the majority of customers go to the worse restaurant?

(f) Show that there are values of p and q that can make the probability you found in part (e) arbitrarily close to 1/2.

2. An investor can allocate their wealth ω between a safe asset with no return (cash), and two risky assets with random returns $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Thus, if the investor invests amounts $a_1 \geq 0$ and $a_2 \geq 0$ with $a_1 + a_2 \leq \omega$ in the two assets, their final wealth would be $\omega + a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. The investor has a strictly increasing, strictly concave, and twice continuously differentiable Bernoulli utility function.

(a) Show that if the returns of the assets are i.i.d. and have a positive expected value, then the investor invests in both assets.

(b) Construct an example in which $\mathbb{E}[\,\mathbf{x}_1\,] > 0$ and $\mathbb{E}[\,\mathbf{x}_2\,] < 0$ and the investor invests in both assets.

(c) Construct an example in which $\mathbb{E}[\,\mathbf{x}_1\,] > 0$ and $\mathbb{E}[\,\mathbf{x}_2\,] > 0$ and the investor only invests in one asset. [*Hint*: For parts (b) and (c) it suffices to construct lotteries with two points in their support.]

3. Consider an individual with wealth ω who faces n possible losses $l_1, l_2, \ldots, l_n$ such that $0 = l_1 < l_2 < \ldots < l_n < \omega$. The probabilities of these losses are $p_1, p_2, \ldots, p_n$, respectively, with $p_i > 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = 1$. An insurance company offers a wide array of insurance policies. The individual can purchase any policy of the form $(m_1, m_2, \ldots, m_n)$ that pays an amount $m_i \in [0, l_i]$ in case loss l_i occurs. The price (or insurance premium) of such a policy is an increasing function of its actuarial cost. That is, it can be written as $c(\sum_{i=1}^n p_i m_i)$, where $c(\,\cdot\,)$ is an increasing function. Show that any expected utility maximizer with a strictly increasing and strictly concave Bernoulli utility function will choose a policy such that the uninsured loss (or deductible) $l_i - m_i$ is the same for all $i = 2, \ldots, n$. Such policies are said to offer "full coverage above the deductible."

4. Consider the lotteries in Table 1. Is the ranking

$$p_1 \succ p_2 \succ p_3 \succ p_4 \succ p_5$$

consistent with the Independence Axiom?

p_1	\$10 for sure
p_2	\$10 with probability 1/3 and \$0 with probability 2/3
p_3	\$5 for sure
p_4	\$5 with probability 1/2 and \$0 with probability 1/2
p_5	\$10 with probability 1/4 and \$0 with probability 3/4

Figure 1 – Lotteries for Problem 2

5. Consider a finite set X. Denote the expectation and variance of a lottery $p \in \Delta X$ by $\mathbb{E}[\,p\,]$ and $\mathbb{V}[\,p\,]$. Suppose that a decision maker maximizes the utility function over lotteries given by

$$U(p) = \mathbb{E}[\,p\,] - \frac{1}{4}\mathbb{V}[\,p\,].$$

Are their preferences consistent with the Independence Axiom?

6. Consider a finite set $X = \{x_1, \dots, x_n\}$ and a complete and transitive relation $\succcurlyeq$ on ΔX satisfying the Independence Axiom. Show that for every $p, q \in \Delta X$ and every $r \in \mathbb{R}^n$ such that both $p + r$ and $q + r$ are lotteries we have that

$$p \succcurlyeq q \iff (p + r) \succcurlyeq (q + r).$$

7. Fix a finite set X and a preference relation $\succcurlyeq$ on X. Show that if U and V are two expected utility representations of $\succcurlyeq$, then there exists numbers $\alpha > 0$ and β such that, for all $p \in \Delta X$,

$$U(p) = \alpha \cdot V(p) + \beta.$$