Econ 9602 – Problem Set 3

Due on 02/05

1. Let $\{\mathbf{x}_i \mid i = 1, 2, \ldots\}$ be a sequence of i.i.d. gambles, $\mathbf{z}_n = \sum_{i=1}^{n} \mathbf{x}_i$ and $\bar{\mathbf{z}}_n = \mathbf{z}_n/n$ for $n = 1, 2, \ldots$. Denote the Aumann-Serrano index of risky aversion of $\mathbf{z}_n$ by R_n, and that of $\bar{\mathbf{z}}_n$ by $\bar{R}_n$.

(a) Show that R_n does not depend on n.

(b) Express $\bar{R}_n$ as a function of R_n, and show that $\lim_{n\to\infty} \bar{R}_n = 0$.

2. Consider an investor with a twice continuously differentiable Bernoulli utility function u with $u' > 0$ and $u'' < 0$. Suppose the investor faces a background risk $\mathbf{z}$ with $\mathbb{E}[\,\mathbf{z}\,] = 0$ and $\mathbb{E}[\,\mathbf{z}^2\,] > 0$. Let $v : \mathbb{R} \to \mathbb{R}$ be given by $v(x) = \mathbb{E}[\,u(x + \mathbf{z})\,]$. Compare the Arrow-Pratt coefficients of risk aversion of u and v.

3. Let $\mathbf{x}$ be a gamble that pays \$110 or $-$\$100, each with probability 1/2. Suppose that a risk averse agent with Bernoulli utility function u rejects the gamble at any initial wealth level. That is, $\mathbb{E}[\,u(\omega + \mathbf{x})\,] < u(\omega)$ for all $\omega \in \mathbb{R}$. Show that the agent would always reject *any* lottery that involves a loss of at least $-$\$1000 with probability at least 1/2.

4. Let Ω be the set of possible states of the world, and let Π be an information partition of Ω for some agent. For each $\omega \in \Omega$, let $\pi(\omega)$ denote the unique block of the partition which includes ω. Say that the agent *knows* event $E \subseteq \Omega$ in state ω if $\pi(\omega) \subseteq E$. Let $K(E)$ denote the set of states where the agent knows E, i.e.,

$$K(E) = \{\omega \in \Omega \mid \pi(\omega) \subseteq E\}\,.$$

(a) Let $\Omega = \{0, 1, 2, 3\}$, $\Pi = \{\{0, 1\}, \{2, 3\}\}$, and $E = \{1, 2, 3\}$. Find $K(E)$.

(b) Show that if the agent know E, then E must be true. That is, $K(E) \subseteq E$.

(c) Show that if the agent know E, then she also knows that she knows E. That is, $K(E) \subseteq K(K(E))$ (and therefore $K(K(E)) = K(E)$).

5. Consider a CARA agent with coefficient of risk aversion equal to $\alpha > 0$. Suppose there are two states x_1 and x_2 with probabilities p and $(1 - p)$, respectively. The agent must choose one of three assets a_1, a_2, and a_3. The returns of the assets are $(0, 100)$, $(20, 75)$, and $(40, 50)$, respectively, where the first number represents the payoff in state x_1 and the second number to the payoff in state x_2.

(a) Show that choosing a_2 is rational.

(b) What can we infer about the beliefs and risk aversion of the agent if she chooses a_2?

(c) Consider the signal structure (S, σ) with $S = \{s_1, s_2\}$, $\sigma(s_1|x_1) = 1 - \epsilon$, and $\sigma(s_1|x_2) = \epsilon$ for some $\epsilon > 0$. How much would the agent be willing to pay to observe the signal before choosing an asset?

(d) How does your answer to (c) vary with respect to ϵ and p? Provide some intuition for this answer.

6. Let p and q be lotteries with finite support X. The *Kullback-Leibler divergence* between p and q is the number $D(p||q)$ defined by

$$D(p||q) = \sum_{x \in X} p(x) \log\left(\frac{p(x)}{q(x)}\right).$$

Which of the properties that define a metric metric are satisfied by the Kullback-Leibler divergence?

7. Suppose the true state of nature is either high or low. Consider an agent that must choose an action $a \in \{0, 1\}$. The agent prefers action a if and only if she believes the state is high with probability at least $1/2$. Suppose that she assigns a prior belief $p \in (0, 1)$ to the state being high. Before choosing an action, the agent observes a signal coming from the structure (S, σ).

(a) Find all the distributions over posteriors that are consistent with the prior beliefs of the agent.

(b) Show that, if $p \leq 1/2$, then there exists a signal structure that induces the agent to choose $a = 1$ with probability $2p$.

(c) Show that there does *not* exists a signal structure that induces the agent to choose $a = 1$ with probability strictly greater than $2p$.