Knowledge and common knowledge

Slides 2 – Rationalizability

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Euclidean spaces

- $\mathbb{R}^n = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_n) \,\middle|\, \mathbf{x}_i \in \mathbb{R} \text{ for } i - 1, \ldots, n \right\}$
- The scalar product of $\mathbf{x} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ is defined as

$$\mu\mathbf{x} = (\mu x_1,\, \mu x_2, \ldots, \mu x_n) \in \mathbb{R}^n$$

- The sum of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1,\, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n$$

- The inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as

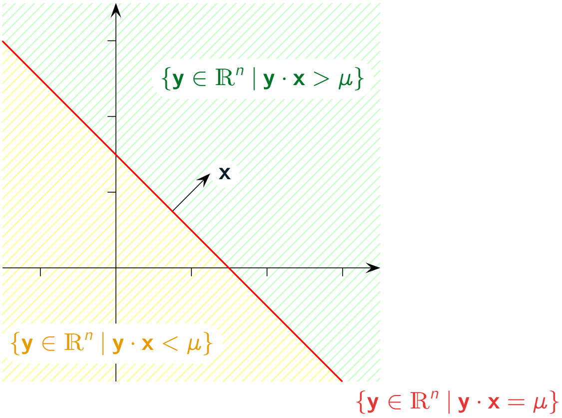
$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$$

- The Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}} \in \mathbb{R}_+$$

Hyperplanes

- A vector $\mathbf{x} \in \mathbb{R}^n$ and a scalar c partition $\mathbb{R}^n$ into two half-spaces and a hyperplane

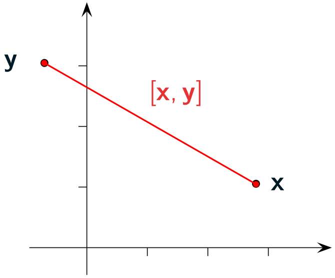


Convex combinations

- $\mathbf{z} \in \mathbb{R}^n$ is a convex combination of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ if there exists a scalar $\mu \in (0, 1)$ such that

$$\mathbf{z} = \mu\mathbf{x} + (1 - \mu)\mathbf{y}$$

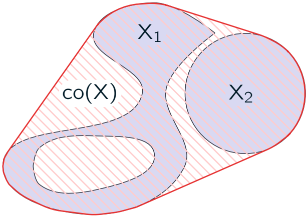
- The set of convex combinations of $\mathbf{x}$ and $\mathbf{y}$ is the line segment joining $\mathbf{x}$ and $\mathbf{y}$, it is denoted by $[\mathbf{x}, \mathbf{y}]$



Convex sets

- A set $\mathbf{X} \subseteq \mathbb{R}^n$ is convex if it contains all the linear combinations of its elements
- The convex hull of a set $\mathbf{X} \subseteq \mathbb{R}^n$ is the smallest convex set $\text{co}(\mathbf{X}) \subseteq \mathbb{R}^n$ containing $\mathbf{X}$
- It is easy to show that the convex hull of $\mathbf{X}$ can be written as

$$\text{co}(\mathbf{X}) = \left\{ \sum_{i=1}^{m} \mu_i \mathbf{x}_i \;\middle|\; \mathbf{x}_i \in \mathbf{X} \quad \mu_i \geq 0 \quad \sum_{i=1}^{m} \mu_i = 1 \right\}$$



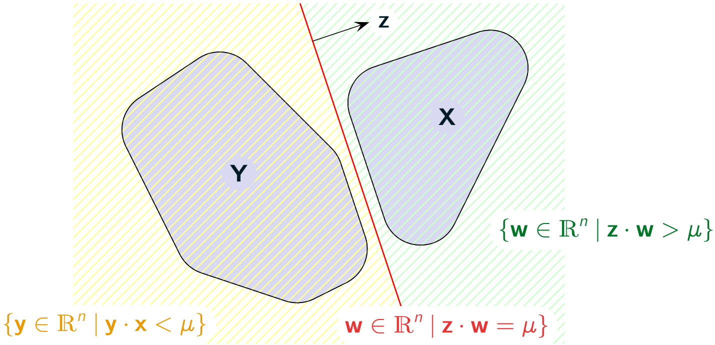
$\mathrm{X} = \mathrm{X}_1 \cup \mathrm{X}_2$

Separating hyperplane theorem

Minkowski's Separating hyperplane theorem

For non-empty convex sets $\mathbf{X}, \mathbf{Y} \subseteq \mathbb{R}^n$ *with disjoint interiors, there exists a non-null vector* $\mathbf{z} \in \mathbb{R}^n \backslash \{0\}$ *and a scalar* $\mu \in \mathbb{R}$ *such that*

$$\sup\{\mathbf{z} \cdot \mathbf{x} \mid \mathbf{x} \in \mathbf{X}\} \leq \mu \leq \inf\{\mathbf{z} \cdot \mathbf{y} \mid \mathbf{y} \in \mathbf{Y}\}$$



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Single agent decision problem

- Ω – finite set of states of the world
- Π – information partition
- K – knowledge operator
- p – prior beliefs, can be thought of as a vector $\mathbf{p} = \big(p(\omega)\big)_{\omega\in\Omega} \in \mathbb{R}^\Omega$
- q – posterior beliefs

- S – finite set of strategies, denoted by s, s', ...
- $u : S \times \Omega \to \mathbb{R}$ – payoffs (preferences)

- $\mathbf{u}(s) = \big(u(s,\omega)\big)_{\omega\in\Omega} \in \mathbb{R}^\Omega$ for $s \in S$, $\quad \mathbf{u}(S) = \{\mathbf{u}(s) \mid s \in S\}$

- Expected utility $U : S \times \Delta(\Omega) \to \mathbb{R}$ is given by:

$$U(s,p) \equiv \mathbb{E}_p\,[\,u(s,\omega)\,] = \int_\Omega u(s,\omega)\,dp(\omega) = \sum_{\omega\in\Omega} u(s,\omega)\,p(\omega) = \mathbf{p}\cdot\mathbf{u}(s)$$

Rationality

- Rational agents choose best responses, i.e. strategies which maximize their expected utility given their beliefs

$$
\begin{aligned}
\mathrm{BR}(p) &= \arg\max \left\{ U(s,p) \,\middle|\, s \in S \right\} \\
&= \left\{ s \in S \,\middle|\, \quad \forall s' \in S\colon\ U(s,p) \geq U(s',p) \right\}
\end{aligned}
$$

- Suppose that we know U but not p
- An action is rationalizable if it is a best response to some beliefs

$$
\mathrm{BR} = \bigcup_{\mathrm{p} \in \Delta(\Omega)} \mathrm{BR}(p)
$$

- Actions which are not rationalizable are called never-best-responses

$$
\mathrm{NBR} = S \backslash \mathrm{BR}
$$

Strictly dominated strategies

- s strictly dominates s' if and only if s is strictly preferred to s' independently of ω, i.e. if and only if

$$\forall \omega \in \Omega \qquad u(s, \omega) > u(s', \omega)$$

- s is strictly dominated by a pure strategy, if there exists some $s' \in S$ which strictly dominates it
- Strictly dominated strategies are never-best-responses
  - Suppose that s strictly dominates s'
  - Then for all $p \in \Delta(\Omega)$:

$$U(s, p) = \sum_{\omega \in \Omega} p(\omega) u(s, \omega) > \sum_{\omega \in \Omega} p(\omega) u(s', \omega) = U(s', p)$$

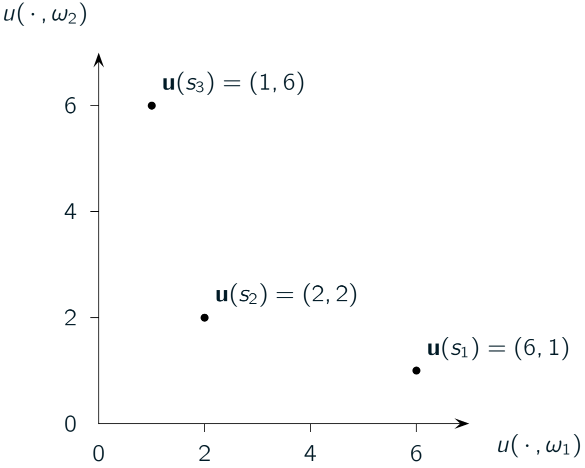
Example: pure dominance and NBR

	ω_1	ω_2
s_1	6	1
s_2	2	2
s_3	1	6

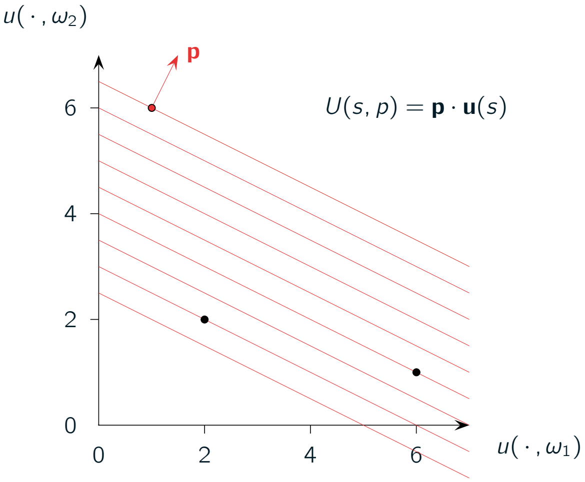
$u(\omega, s)$

- If $p(\omega_1) > 1/2$ then the unique best response is s_1.
- If $p(\omega_1) < 1/2$ then the unique best response is s_3.
- If $p(\omega_1) = 1/2$ then both s_1 and s_3 generate an expected utility of 3.5, while s_2 generates an expected utility of 2.
- Hence s_2 is a NBR despite the fact that it is not strictly dominated by any pure strategy

Example: pure dominance and NBR

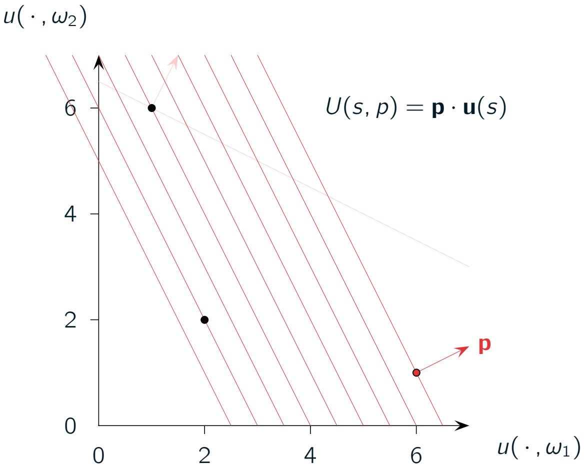


Example: pure dominance and NBR



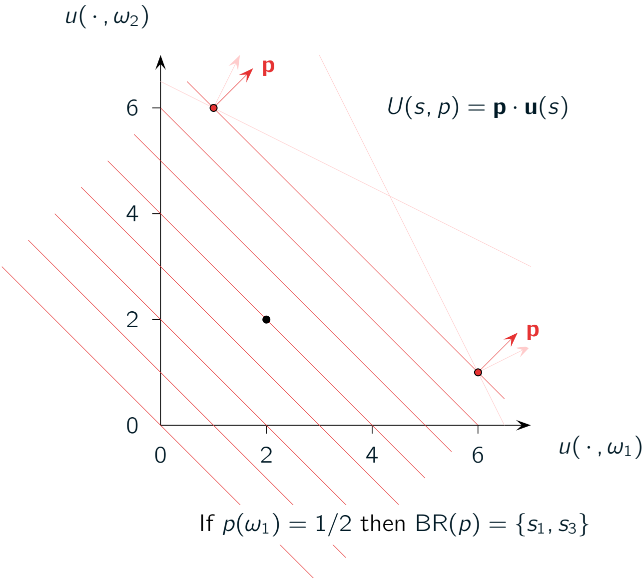
If $p(\omega_1) < 1/2$ then s_3 is the unique best response

Example: pure dominance and NBR



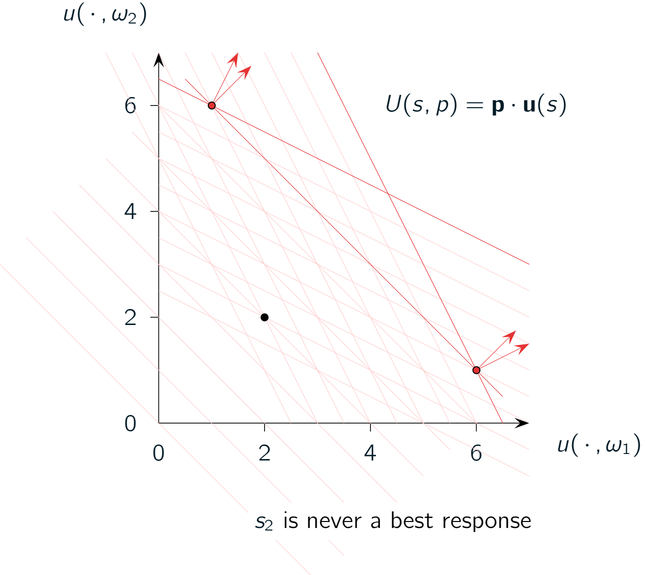
If $p(\omega_1) > 1/2$ then s_1 is the unique best response

Example: pure dominance and NBR



If $p(\omega_1) = 1/2$ then $\text{BR}(p) = \{s_1, s_3\}$

Example: pure dominance and NBR



s_2 is never a best response

Strict dominance by mixed strategies

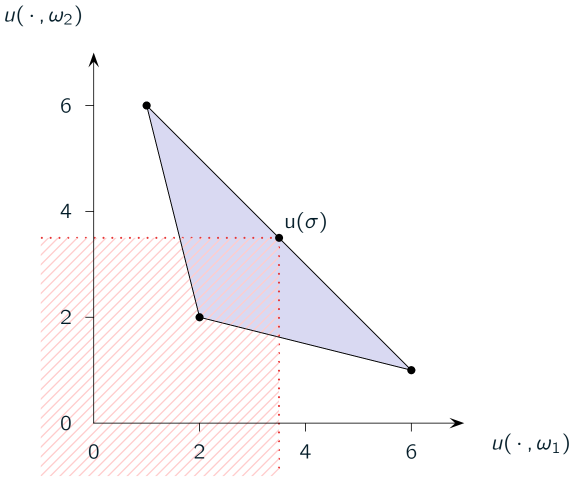
- A mixed strategy is a distribution $\sigma \in \Delta(S)$
  - Could be thought of as actual randomization
  - It is useful just as a tool to characterize NBR
  - By mixing, a player can guarantee any expected payoff vector $\mathbf{u} \in \text{co}(\mathbf{u}(S))$
- A mixed strategy σ strictly dominates s if and only if:

$$\forall \omega \in \Omega \qquad u(s,\omega) < \sum_{s' \in S} u(s',\omega) \cdot \sigma(s')$$

Theorem (Pearce,1984)

A pure strategy is a never-best-response if and only if it is strictly dominated by a pure or mixed strategy.

Example: pure dominance and NBR



s_2 is dominated by the mixed strategy $\sigma = \frac{1}{2}[s_1] + \frac{1}{2}[s_3]$

Strict dominance and rationalizability (proof)

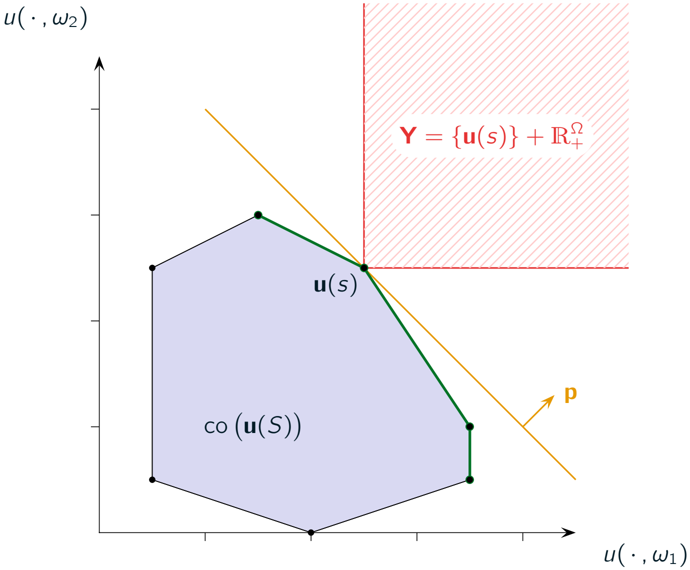
$\Leftarrow$

- Suppose that s is strictly dominated by σ
- This implies that for every $p \in \Delta(\Omega)$:

$$
\begin{aligned}
U(s,p) &= \sum_{\omega\in\Omega} p(\omega)u(s,\omega) \\
&< \sum_{\omega\in\Omega} p(\omega)\left(\sum_{s'\in S}\sigma(s')u(s',\omega)\right) \\
&= \sum_{\omega'\in S}\sigma(s')\left(\sum_{\omega\in W} p(\omega)u(s',\omega)\right) \\
&= \sum_{\omega'\in S}\sigma(s')U(s',p)
\end{aligned}
$$

- And therefore there exists some $s' \in S$ such that $U(s,p) < U(s',p)$
- Hence we have $s \notin \mathrm{BR}(p)$ for any $p \in \Delta(\Omega)$, i.e. $s \notin \mathrm{BR}$

Strict dominance and rationalizability (proof)



Strict dominance and rationalizability (proof)

$\Rightarrow$

- Let $\mathbf{Y} \equiv \{\mathbf{u}(s)\} + \mathbb{R}_+ = \{\mathbf{y} \in \mathbb{R}^\Omega \mid \forall \omega \in \Omega : \mathbf{y}_\omega \geq u(s,\omega)\}$
- If s is not strictly dominated, then there is no $\mathbf{x} \in \text{co}(\mathbf{u}(S))$ such that $\mathbf{x}_\omega > u(s,\omega)$ for all $\omega \in \Omega$
- This implies that the interiors of $\mathbf{Y}$ and $\text{co}\left(\mathbf{u}(S)\right)$ are disjoint
- By the SHT, there exists some $\mathbf{p} \in \mathbb{R}^\Omega \setminus \{0\}$ such that

$$\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{y} \quad \text{for} \quad \mathbf{x} \in \text{co}(\mathbf{u}(S)) \quad \text{and} \quad \mathbf{y} \in \mathbf{Y} \tag{1}$$

- Since $\mathbf{u}(s) \in \mathbf{Y}$, if we can show $\mathbf{p} \in \Delta(\Omega)$ we would have $s \in \text{BR}(\mathbf{p}) \subseteq \text{R}$, completing the proof
- Without loss of generality (why?) we can assume $\sum_{\omega \in \Omega} \mathbf{p}_\omega = 1$
- Suppose towards a contradiction that $\mathbf{p}_{\omega_0} < 0$ for some $\omega_0 \in \Omega$
  - Let $\mathbf{z} \in \mathbb{R}^\Omega$ be the vector given by $\mathbf{z}_{\omega_0} = u(s,\omega) + 1$, and $\mathbf{z}_\omega = u(s,\omega)$ if $\omega \neq \omega_0$
  - However $\mathbf{p} \cdot \mathbf{z} = \mathbf{p} \cdot \mathbf{u}(s) + \mathbf{p}_{\omega_0} < \mathbf{p} \cdot \mathbf{u}(s)$
  - Since $\mathbf{u}(s) \in \text{co}\left(\mathbf{u}(S)\right)$ and $\mathbf{z} \in \mathbf{Y}$, this contradicts (1)

QED

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Strategic form game of complete information

- A strategic form game is a tuple $G = (I, S, u)$
  - $I = \{1, 2, \ldots, n\}$ – players
  - $S_i = \{s_i, s_i, \ldots\}$ – finite set of i's strategies
  - $S = \{s, s', \ldots\} \equiv \underset{i \in I}{\times} S_i$ – set of strategy profiles
    - $S_{-i} = \{s_{-i}, s'_{-i}, \ldots\} \equiv \underset{j \in I \setminus \{i\}}{\times} S_j$
    - $s = (s_i, s_{-i})$
  - $u_i : S \to \mathbb{R}$ – utility function

Rationalizability

- **Complete information** – the game (I, S, u) is common knowledge
- **Imperfect information** – players may be uncertain about the choices and beliefs of their opponents

Question

What predictions can we make if we only assume that all players are rational, and that this fact is common knowledge?

Example: a 2×2 game

	a	b
A	100 , 100	0 , 95
B	75 , 80	75 , 75

- The row player could choose A if he/she believed $\Pr(s_2 = a) \geq 3/4$
- The row player could choose B if he/she believed $\Pr(s_2 = a) \leq 3/4$
- If the column player were rational he/she would choose a
- Hence, if the row player knew that the column player is rational, he would choose A

Example: weak dominance

	a	b
A	100 , 100	0 , 100
B	75 , 80	75 , 75

- Now the column player could very well choose b
- This would require absolute certainty that the row player is choosing A

Games as simultaneous decision problems

- Think of G as n simultaneous single agent problems
  - Each agent i must choose $s_i \in S_i$
  - being uncertain of a state $s_{-i} \in S_{-i}$ describing his opponent's choices
  - His/her payoffs are given by $u_i : S_i \times S_{-i} \to \mathbb{R}$
  - His/her beliefs can be described by $p_i \in \Delta(S_{-i})$

Rationality in strategic form games

- As before, expected utility is given by

$$U_i(s_i, p_i) = \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) p_i(s_{-i})$$

- Rationality means that the choice of i should satisfy

$$s_i \in \mathrm{BR}(p_i) = \left\{ s_i' \in S_i \;\middle|\; \forall s_i'' \in S_i \quad U_i(s_i', p_i) \geq U_i(s_i'', p_i) \right\}$$

- Assuming common knowledge of rationality allows to impose additional restrictions on p_i

Complete justifications

- Consider a game between Anna and Bob, and assume common knowledge of rationality
- Anna may consider playing some strategy s_{Anna} only if it is a best response to some belief $p_{\text{Anna}} \in \Delta(S_{\text{Bob}})$
- Since Anna knows that Bob, is rational, she must be able to justify this belief, i.e. if $p_{\text{Anna}}(s_{\text{Bob}}) > 0$ then there must exist some belief $p_{\text{Bob}} \in \Delta(S_{\text{Anna}})$ for which s_{Bob} is a best response
- Similarly, since Anna knows that Bob knows that she is rational, if $p_{\text{Bob}}(s'_{\text{Anna}}) > 0$ there must exist some belief $p'_{\text{Anna}} \in \Delta(S_{\text{Bob}})$ for which s'_{Anna} is a best response
- Continuing this argument, since the game is finite, we must eventually reach a cycle!
- We need a set of strategy profiles $T = \times_{i \in I} T_i \subseteq S$ such that for every $i \in I$, every $s_i \in T_i$ can be justified as a best response to strategies in T_{-i}

Ratonalizability in terms of self generation

- Say that s_i is rational for i with respect to $T_{-i} \subseteq S_{-i}$ if it is a best response to some belief *which assigns full probability to* T_{-i}
- Let $\mathrm{BR}_i(T_{-i})$ denote the set of such actions:

$$\mathrm{BR}_i(T_{-i}) = \left\{ s_i \in S_i \;\middle|\; \left(\exists p_i \in \Delta(S_i)\right)\left(p_i(T_{-i}) = 1 \;\wedge\; s_i \in \mathrm{BR}_i(p_i)\right) \right\}$$

- A set of strategy profiles $T = \times_{i \in I} T_i \subseteq S$ is self-rationalizable if and only if for every player $i \in I$ every strategy $s_i \in T_i$ is rational for i with respect to T_{-i}, i.e. if and only if:

$$T \subseteq \mathrm{BR}(T) = \bigtimes_{i \in I} \mathrm{BR}_i(T_{-i})$$

- The set of rationalizable strategies R is defined to be the largest self-rationalizable set, i.e.

$$\mathrm{R} = \bigcup \left\{ T \subseteq S \;\middle|\; T \subseteq \mathrm{BR}(T) \right\}$$

Some observations

Lemma

The correspondence of rational actions $\mathrm{BR}_i : 2^{S_{-i}} \to 2^{S_i}$ *is* $\subseteq$*-monotone, i.e.* $\mathrm{BR}_i(T_{-i}) \subseteq \mathrm{BR}_i(T'_{-i})$ *whenever* $T_{-i} \subseteq T'_{-i}$

- Suppose that $T_{-i} \subseteq T'_{-i}$ and $s_i \in \mathrm{BR}_i(T_{-i})$
- Then s_i is a best response to some $p_i \in \Delta(S_{-i})$ with $p_i(T_{-i}) = 1$
- Since $T_{-i} \subseteq T'_{-i}$ and beliefs are sub-additive, we have $p_i(T'_{-i}) = 1$ and hence $s_i \in \mathrm{BR}_i(T'_{-i})$

QED

Corollary

The set of rationalizable actions is self-rationalizable

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Informal argument

1. Everybody is rational
   - ⇒ players will choose best responses to *arbitrary* beliefs
2. Everybody knows that everybody is rational
   - ⇒ players believe that their opponents will play best responses
   - ⇒ they will choose best responses to best responses
3. Everybody knows that everybody knows that everybody is rational
   - ⇒ players believe that their opponents will play best responses to best responses
   - ⇒ they will choose best responses to best responses to best responses

   ⋮

4. There is common knowledge of rationality
   - ⇒ players will choose rationalizable strategies

Never-best responses and dominance

- Previous argument suggests that rationalizable strategies arise from the successive elimination of never best responses $\text{NBR}_i(T_{-i}) = S_i \backslash \text{BR}_i(T_{-i})$
- A mixed strategy $\sigma_i \in \Delta(S_i)$ strictly dominates $s_i \in S_i$ with respect to T_{-i}, if and only if

$$\forall s_{-i} \in T_{-i} \qquad u_i(s_i, s_{-i}) < \sum_{s_i' \in S_i} \sigma_i(s_i') u_i(s_i', s_{-i})$$

- Result for single agent decision problems implies that $\text{NBR}_i(T_{-i})$ corresponds to the set of dominated strategies
- Side note – if we only assume common knowledge of ordinal preferences, then we only have to consider dominance by pure strategies (Borgers, 1993)

Iterated dominance

- The process of iterative elimination of (all) strictly dominated strategies is described by the sequence $(S^n = \times_{i\in I} S_i^n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ defined iteratively as follows

$$S^1 = S$$

$$\forall n \geq 1 \quad S_i^{n+1} = S_i \backslash \mathrm{NBR}(S_{-i}^n) = \mathrm{BR}_i(S_{-i}^n)$$

$$\forall n \geq 1 \quad S^{n+1} = \mathop{\times}_{i\in I} S_i^{n+1}$$

- The set of strategies surviving the iterated removal of strictly dominated strategies is:

$$S^\infty \equiv \bigcap \left\{ S^n \;\middle|\; n \in \mathbb{N} \right\}$$

- We will show that it corresponds to the set of rationalizable strategies, i.e. $S^\infty = \mathrm{R}$

Example: A 4×4 game

	a	b	c	d
w	0 , 7	2 , 5	7 , 0	0 , 1
x	5 , 2	3 , 3	5 , 2	0 , 1
y	7 , 0	2 , 5	0 , 7	0 , 1
z	0 , 0	0 , -2	0 , 0	10 , -1

Example: A 4×4 game

	a	b	c	d
w	0 , 7	2 , 5	7 , 0	0 , 1
x	5 , 2	3 , 3	5 , 2	0 , 1
y	7 , 0	2 , 5	0 , 7	0 , 1
z	0 , 0	0 , −2	0 , 0	10 , −1

d is strictly dominated by the mixed strategy $(1/2, 0, 1/2, 0)$ that mixes a and c with equal probabilities

Example: A 4×4 game

	a	b	c	d
w	0 , 7	2 , 5	7 , 0	0 , 1
x	5 , 2	3 , 3	5 , 2	0 , 1
y	7 , 0	2 , 5	0 , 7	0 , 1
z	0 , 0	0 , −2	0 , 0	10 , −1

z is not dominated on the first stage but it is dominated by x once we eliminate d

Example: A 4×4 game

	a	b	c	d
w	0 , 7	2 , 5	7 , 0	0 , 1
x	5 , 2	3 , 3	5 , 2	0 , 1
y	7 , 0	2 , 5	0 , 7	0 , 1
z	0 , 0	0 , -2	0 , 0	10 , -1

w can be rationalized by 1 using the following argument:

- 1 believes that 2 will choose c
- 1 believes that 2 believes that 1 will choose y
- 1 believes that 2 believes that 1 believes that 2 will choose a
- 1 believes that 2 believes that 1 believes that 2 believes that 1 will choose w . . .

Iterated dominance

Theorem (Pearce,1984)

The iterated removal of strictly dominated strategies converges to the set of rationalizable strategy profiles.

- $\mathrm{R} \subseteq S^\infty$
  - Since $\mathrm{R} \subseteq \mathrm{BR}(\mathrm{R})$, and BR is $\subseteq$-monotone, it follows that $\mathrm{R} \subseteq \mathrm{BR}(T)$ whenever $\mathrm{R} \subseteq T$
  - Since $\mathrm{R} \subseteq S$, it follows that $\mathrm{R} \subseteq S^1$
  - If $\mathrm{R} \subseteq S^n$, then $\mathrm{R} \subseteq S^{n+1}$
  - By the principle of mathematical induction, $\mathrm{R} \subseteq S^n$ for all $n \in \mathbb{N}$
  - Therefore $\mathrm{R} \subseteq \cap_{n \in \mathbb{N}} S^n = S^\infty$

Iterated dominance

Theorem (Pearce,1984)

The iterated removal of strictly dominated strategies converges to the set of rationalizable strategy profiles.

- $\mathrm{R} \subseteq S^{\infty}$
- $S^{\infty} \subseteq \mathrm{R}$
  - First we will show that S^n is a decreasing sequence
    - By construction $S^2 \subseteq S^1$
    - Suppose $S^{n+1} \subseteq S^n$, monotonicity of BR implies that $S^{(n+1)+1} = \mathrm{BR}(S^{n+1}) \subseteq \mathrm{BR}(S^n) = S^{n+1}$
    - By induction this implies $S^{n+1} \subseteq S^n$ for all $n \in \mathbb{N}$
  - Hence, since the game is finite, there exists some n such that $S^{n+1} = \mathrm{BR}(S^n) \subseteq S^n$
  - Which implies that $S^n \subseteq \mathrm{R}$
  - And therefore $S^{\infty} = \cap_{n \in \mathbb{N}} S^n \subseteq S^n \subseteq \mathrm{R}$

$\mathcal{QED}$

Existence

Proposition

The set of rationalizable strategies ins non-empty.

- The best response correspondence is nonempty-valued, i.e. $\mathrm{BR}_i(T_{-i}) \neq \emptyset$ whenever $T_{-i} \neq \emptyset$
  - Fix some $s_{-i} \in T_{-i}$ and pick some arbitrary $s_i \in S_i$
  - If $s_i \in \mathrm{BR}_i(\{s_{-i}\})$ then $s_i \in \mathrm{BR}_i(T_{-i})$ and we are done
  - Otherwise, there exists $s_i' \in S_i \backslash \{s_i\}$ such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$
  - If $s_i' \in \mathrm{BR}_i(\{s_{-i}\})$ we are done
  - Otherwise, there exists $s_i'' \in S_i \backslash \{s_i, s_i'\}$ such that $u_i(s_i'', s_{-i}) > u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$
  - Since S_i is finite, the process will stop at some best response to $\{s_{-i}\} \subseteq T_{-i}$

Existence

Proposition

The set of rationalizable strategies ins non-empty.

- The best response correspondence is nonempty-valued, i.e. $\mathrm{BR}_i(T_{-i}) \neq \emptyset$ whenever $T_{-i} \neq \emptyset$
- This implies that $S^1 = \mathrm{BR}(S) \neq \emptyset$, and if $S^{n+1} = \mathrm{BR}(S^n) \neq \emptyset$ whenever $S^n \neq \emptyset$
- By induction this implies that $S^n \neq \emptyset$ for all $n \in \mathbb{N}$
- Since the elimination process converges infinite time to R, this implies that $\mathrm{R} \neq \emptyset$

QED

Order independence

- We analyzed the successive elimination of all dominated strategies
- What happens if we eliminate some but not all strategies at each step?

- A dominance-elimination procedure is a sequence $(S^n) \in S^{\mathbb{N}}$ such that:
  1. Strategies are eliminated, i.e. $S^{n+1} \subseteq S^n$ for all $n \in \mathbb{N}$
  2. Only dominated strategies are eliminated, i.e. $s_i \notin BR(S^n_{-i})$ whenever $s_i \in S_i^n \backslash S_i^{n+1}$
  3. Whenever there are dominated strategies something is eliminated, i.e. if $S^n \neq BR(S^n)$ then $S^{n+1} \neq S^n$

Proposition

Every dominance elimination procedure converges to the set of rationalizable strategies.

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

The only dominated strategy is a, which is dominated by d

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

After eliminating a, A is dominated by B, and D is dominated by C

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

At this point, d is dominated by b

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

At this point, C is dominated by B

Example: A 4×4 game

	a	b	c	d
A	88 , 78	78 , 91	75 , 85	72 , 85
B	72 , 72	81 , 78	78 , 75	75 , 75
C	94 , 72	78 , 78	75 , 88	88 , 75
D	100 , 88	75 , 81	72 , 78	85 , 97

Finally, c is dominated by b

[0]

A glimpse of universal type spaces

- With complete but imperfect information there is only strategic uncertainty about $\Omega^1 = S$
- First order beliefs belong to $B^0 = \times_{i \in I} \Delta(\Omega^0)$
- Players may also be uncertain about first order beliefs, let $\Omega^2 = \Omega^1 \times B^1$
- Second order beliefs belong to $B^2 = \times_{i \in I} \Delta(\Omega^2)$
- Players may also be uncertain about second order beliefs, let $\Omega^3 = \Omega^2 \times B^2 = S \times B^1 \times B^2$
- Third order beliefs belong to $B^3 = \times_{i \in I} \Delta(\Omega^3)$

  ...
- A belief hierarchy is a vector $b = (b_n)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} B^n = \times_{n \in \mathbb{N}} \times_{i \in I} \Delta(\Omega^n)$ satisfying some consistency conditions
- The same information can be captured by our simple espitemic models provided they are "large" enough
- This insight was suggested by Harnasyi (1967) and formalized by Mertens & Zamir (1985)

Epistemic model

- Ω – states of the world
- Π_i – Information partition
- K_i – Knowledge operator
- p_i – Priors beliefs
- q_i – Posterior beliefs
- $c_i : \Omega \to S_i$ – Choices
  - Measurable with respect to Π_i, i.e. $c_i(\omega) = c_i(\omega')$ whenever $\pi_i(\omega) = \pi_i(\omega')$
- $u_i : S \times \Omega \to \mathbb{R}$ – ex-post utility
  - Derived from the strategic form game, i.e. $u_i(s_i, \omega) = u_i\big(s_i, c_{-i}(\omega)\big)$

Expected utility

- Ex-ante expected utility

$$
\begin{aligned}
U_i(s_i) &\equiv \mathbb{E}_{p_i}\left[\, u_i(s_i, \omega) \,\right] \\
&= \int_{\Omega} u_i(s_i, \omega)\, dp_i(\omega) = \sum_{\omega \in \Omega} u(s, \omega_i)\, p_i(\omega)
\end{aligned}
$$

- Interim expected utility

$$
\begin{aligned}
U_i(s_i | \pi_i) &\equiv \mathbb{E}_{q_i(\,\cdot\,|\pi_i)}\left[\, u(s_i, \omega) \,\right] \\
&= \int_{\Omega} u_i(s_i, \omega)\, dq_i(\omega | \pi_i) = \sum_{\omega \in \Omega} u_i(s_i, \omega)\, q_i(\omega | \pi_i)
\end{aligned}
$$

Interim vs. ex-ante rationality

Proposition

c_i maximizes ex-ante expected utility if and only if it maximizes interim expected utility almost surely (with prior probability 1)

- Notice that

$$\begin{aligned} U_i(c_i) &= \sum_{\omega \in \Omega} u(c_i(\omega), \omega)\, p_i(\omega) \\ &= \sum_{\pi_i \in \Pi_i} \sum_{\omega \in \pi_i} u(c_i(\omega), \omega)\, p_i(\pi_i) q_i(\omega|\pi_i) = \sum_{\pi_i \in \Pi_i} p_i(\pi_i) U_i(c_i(\pi_i)|\pi_i) \end{aligned}$$

- Therefore $U_i(c_i) \geq U_i(c_i')$ for all c_i' if and only if $U_i(c_i(\pi_i)|\pi_i) \geq U_i(s_i|\pi_i)$ for all s_i and all π_i with $p_i(\pi_i) > 0$

$\mathcal{QED}$

Common knowledge of rationality

- Derive first order beliefs about choices, from beliefs about states:

$$q_i(s_{-i}|\omega) = q_i(\{\omega' \in \Omega \mid c_{-i}(\omega') = s_{-i}\}|\omega)$$

- Write expected utility in terms of first order beliefs:

$$U_i(s_i|\omega) = \sum_{s_i \in S_{-i}} u_i(s_i, s_{-i}) q_i(s_{-i}|\omega)$$

- i is rational at ω if his choice $(c_i(\omega))$ is optimal given his beliefs
- The set of states where i is rational is thus:

$$\text{RAT}_i = \left\{\omega \in \Omega \middle| \quad \forall s_i \in S_i \; U_i(c_i(\omega)|\omega) \geq U_i(s_i|\omega)\right\}$$

- The set of states where everyone is rational is this $\text{RAT} = \times_i \text{RAT}_i$
- There is common knowledg eof rationality if and only if RAT is common knowledge

Rationalizability and common knowledge of rationality

Theorem

A strategy profile s^* is rationalizable if and only if there exists a model and a state ω^* such that there is common knowledge of rationality at ω^* and $c(\omega^*) = s^*$.

$\Rightarrow$)

- Suppose $\omega^* \in \mathrm{K}_\mathrm{I}^\infty(\mathrm{RAT})$ and $c(\omega^*) = s$
- There exists $E \subseteq \Omega$ such that $E = \mathrm{K}_\mathrm{I}(E)$ and $\omega^* \in E \subseteq \mathrm{RAT}$
- $T_i = \{c_i(\omega) \mid \omega \in E\}$, strategies chosen by i somewhere in E
- For $\omega \in W$
  - Since $E = \mathrm{K}_\mathrm{I}(E)$, $\pi_i(\omega) \subseteq E$ and hence $q_i(T_{-i}|\omega) = 1$
  - Since $E \subseteq \mathrm{RAT}_i$, $c_i(\omega)$ maximizes $U_i(\cdot|\pi_i(\omega))$
  - Therefore $c_i(\omega) \in \mathrm{BR}_i(T_{-i})$
- Hence $T \subseteq \mathrm{BR}(T)$ and, by definition $T \subseteq \mathrm{R}$
- Since $\omega^* \in T$, this implies that $\omega^* \in \mathrm{R}$

Example: A 4×4 game

	a	b	c	d
w	0 , 7	2 , 5	7 , 0	0 , 1
x	5 , 2	3 , 3	5 , 2	0 , 1
y	7 , 0	2 , 5	0 , 7	0 , 1
z	0 , 0	0 , -2	0 , 0	10 , -1

Example: A 4×4 game

Epistemic model

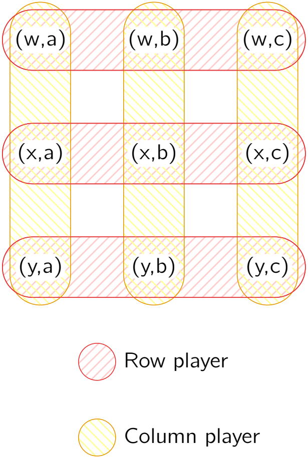
(w,a)	(w,b)	(w,c)
(x,a)	(x,b)	(x,c)
(y,a)	(y,b)	(y,c)

$\Omega = \mathrm{R}$

$c(\omega) = \omega$

Example: A 4×4 game

Epistemic model



$$\Pi_1 = \{\{s_1\} \times R_2 \mid s_1 \in R_1\}$$
$$p_1\big((w,a)\big) = 1$$
$$q_1\big((w,a)|w\big) = 1$$
$$q_1\big((x,b)|x\big) = 1$$
$$q_1\big((y,c)|y\big) = 1$$

$$\Pi_2 = \{\{s_2\} \times R_1 \mid s_2 \in R_2\}$$
$$p_2\big((y,a)\big) = 1$$
$$q_2\big((y,a)|a\big) = 1$$
$$q_2\big((x,b)|b\big) = 1$$
$$q_2\big((w,c)|c\big) = 1$$

$$\mathrm{RAT} = K_I^{\infty}(\mathrm{RAT}) = \Omega$$

Rationalizability and common knowledge of rationality

Theorem

A strategy profile s^* is rationalizable if and only if there exists a model and a state ω^* such that there is common knowledge of rationality at ω^* and $c(\omega^*) = s^*$.

$\Leftarrow$)

- Suppose $s^* \in \mathrm{R}$
- Since $\mathrm{R} = \mathrm{BR}(\mathrm{R})$, for $i \in I$ and $s_i \in \mathrm{R}_i$ there exists $p_{i,s_i} \in \Delta(S_{-i})$ such that $p_i^*(\mathrm{R}_{-i}) = 1$ and $s_i \in \mathrm{BR}_i(p_{i,s_i})$
- Consider the model (Ω, Π, p, q, c) with:
  - $\Omega = \mathrm{R}$, $c_i(\omega) = \omega$
  - $\Pi_i = \big\{\{s_i\} \times \mathrm{R}_{-i} \;\big|\; s_i \in \mathrm{R}_i\big\}$
  - $p_i(s_i^*, s_{-i}) = p_{i,s_i^*}(s_{-i})$
  - $q_i\big(s_{-i} | \{s_i\} \times \mathrm{R}_{-i}\big) = p_{i,s_i}(s_{-i})$
- By construction, $\mathrm{RAT} = \Omega$ and therefore $\mathrm{K}_\mathrm{I}^\infty(\mathrm{RAT}) = \Omega$

QED

[0]

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Cournot competition

- Two firms indexed by $i \in I = \{1, 2\}$ sell the same commodity
- Firms simultaneously choose quantities in $Q_i \in [0, \bar{Q}]$
- The price is determined by the market according to the inverse demand function:

$$P(Q) = P_0 - \sum_{i \in I} Q_i$$

- Firms have constant marginal cost $k \in \mathbb{R}_{++}$ so that their profits are given by:

$$u_i(Q) = \Big(P(Q) - k\Big)Q_i = -Q_i^2 + (P_0 - k - Q_{-i})Q_i$$

Rationalizable quantities

- Given beliefs $p_i \in \Delta\big([0, \bar{Q}]\big)$, let $\tilde{Q}_{-i} = \mathbb{E}_{p_i}[\, Q_{-i}\,]$
- i's best response to p_i is given by:

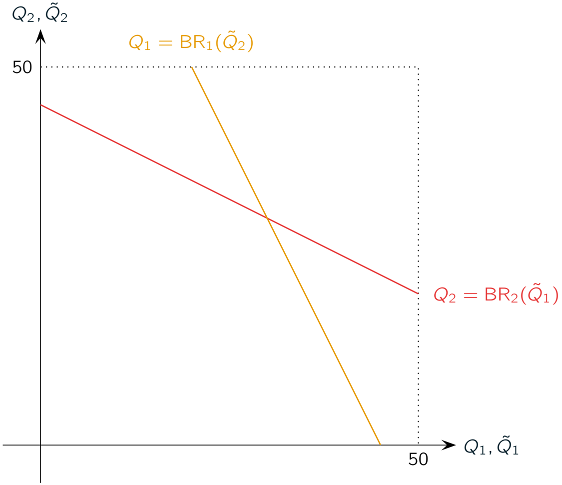
$$\mathrm{BR}_i(p_i) = \frac{1}{2}\Big(P_0 - k - \tilde{Q}_{-i}\Big)$$

Proposition

The unique rationalizable strategy for firm i is $Q_i^* = \frac{1}{3}(P_0 - k)$.

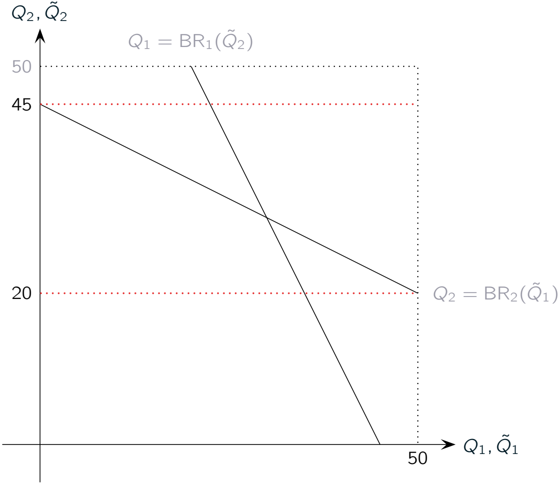
Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

Best response function



Example: $P_0 = 100$, $\bar{Q} = 50$, $k = 10$

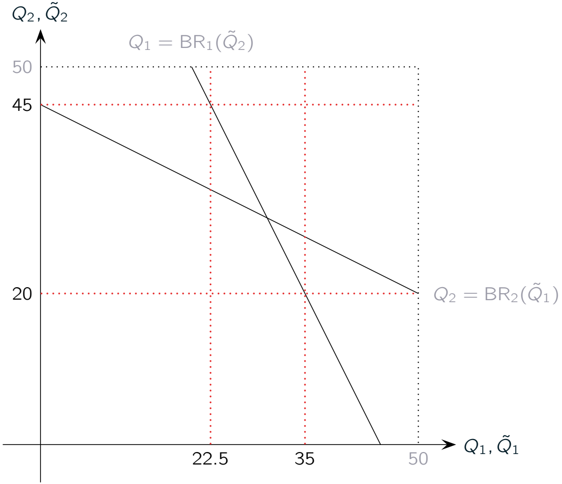
Rationalizability



Firm 2's best response function only takes values between 20 and 45

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

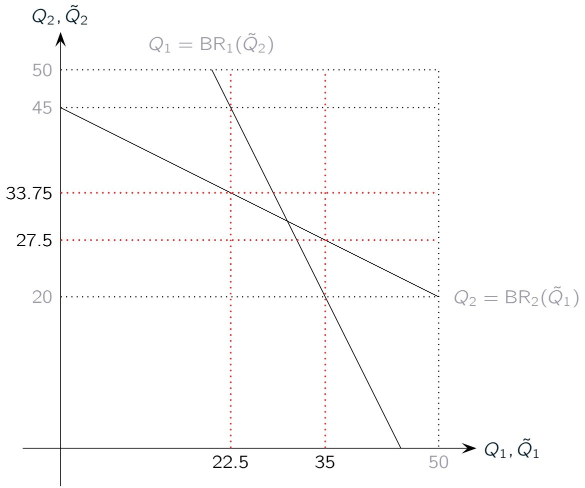
Rationalizability



Knowing that firm 2 will choose a quantity between 20 and 45, firm 1 will only consider choosing quantities between $\mathrm{BR}_2(20) = 35$ and $\mathrm{BR}_2(45) = 22.5$

Example: $P_0 = 100$, $\bar{Q} = 50$, $k = 10$

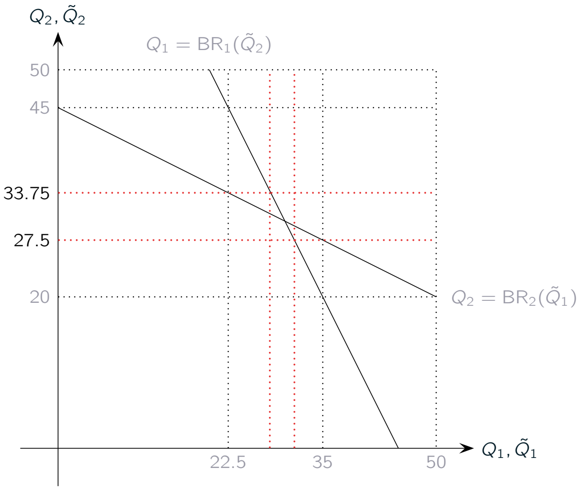
Rationalizability



Knowing that firm 1 will choose a quantity between 22.5 and 35, firm 2 will only consider choosing quantities between $\text{BR}_1(22.5) = 33.75$ and $\text{BR}_1(35) = 27.5$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

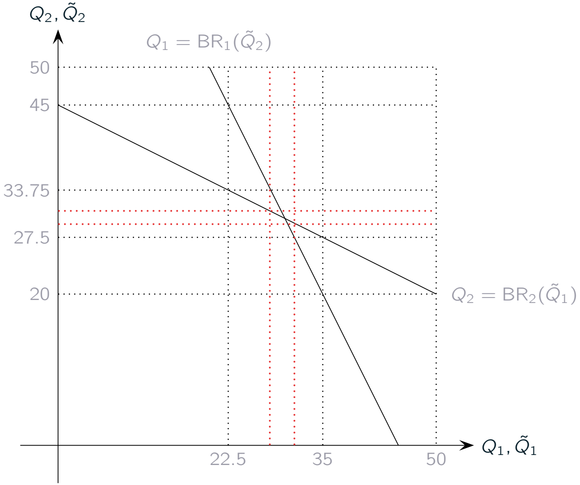
Rationalizability



Knowing that firm 2 will choose a quantity between 27.5 and 33.75, firm 1 will only consider choosing quantities between $\mathrm{BR}_2(27.5) = 31.25$ and $\mathrm{BR}_2(33.75) = 28.125$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

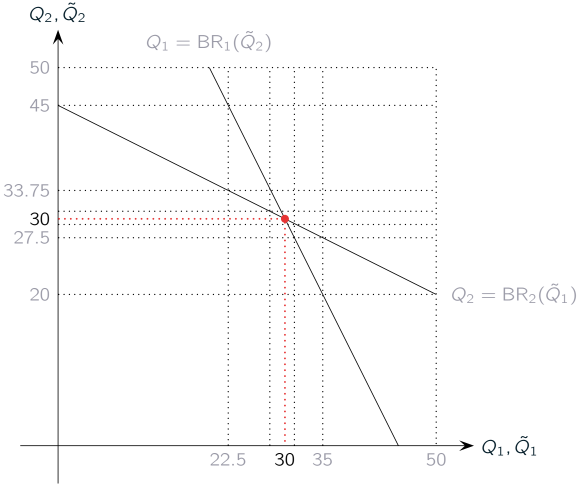
Rationalizability



Knowing that firm 1 will choose a quantity between 28.125 and 33.75, firm 2 will only consider choosing quantities between $\text{BR}_1(28.125) = 30.9375$ and $\text{BR}_1(31.25) = 29.375$

Example: $P_0 = 100, \; \bar{Q} = 50, \; k = 10$

Rationalizability



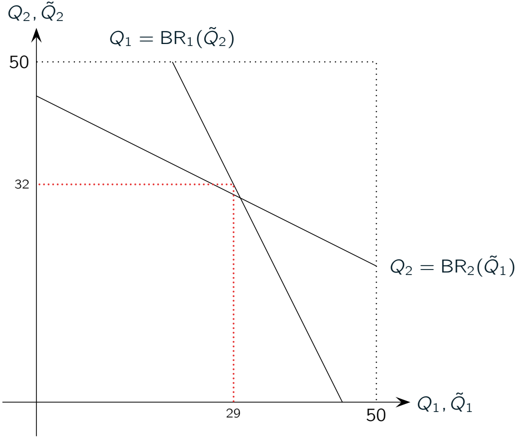
If we carried out this process we will end up concluding that the only rationalizable strategy for each firm is $Q_i = 30$

Example: $P_0 = 100$, $\bar{Q} = 50$, $k = 10$

- Firm 1's best response function only takes values between 20 and 45
- Knowing that firm 1 will choose a quantity between 20 and 45, firm 2 will only consider choosing quantities between $\text{BR}_2(20) = 35$ and $\text{BR}_2(45) = 22.5$
- Knowing that firm 2 will choose a quantity between 22.5 and 35, firm 2 will only consider choosing quantities between $\text{BR}_1(22.5) = 33.75$ and $\text{BR}_1(35) = 27.5$
- Knowing that firm 2 will choose a quantity between 22.5 and 35, firm 2 will only consider choosing quantities between $\text{BR}_2(27.5) = 31.25$ and $\text{BR}_2(33.75) = 27.5$
- Knowing that firm 2 will choose a quantity between 22.5 and 35, firm 2 will only consider choosing quantities between $\text{BR}_1(22.5) = 33.75$ and $\text{BR}_1(35) = 27.5$
- If we carried out this process we will end up concluding that the only rationalizable strategy for each firm is $Q_i = 30$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

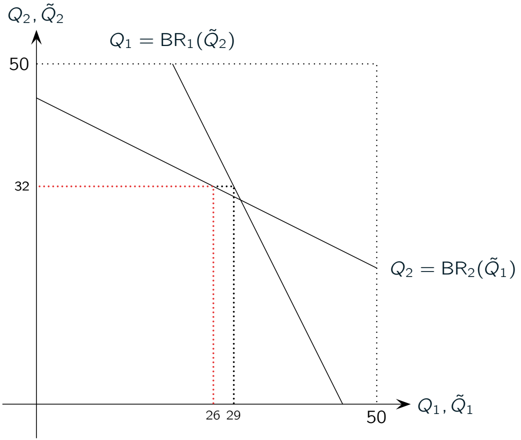
Attempt at rationalizing $Q_i \neq 30$



For firm 1 to choose $Q_1 = 29$, it must believe that firm 2's average quantity is $\tilde{Q}_2 = 32$

Example: $P_0 = 100$, $\bar{Q} = 50$, $k = 10$

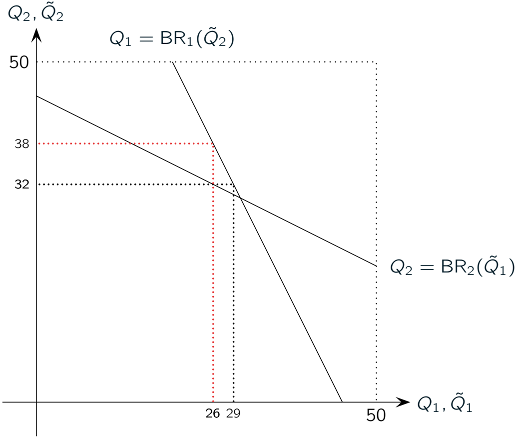
Attempt at rationalizing $Q_i \neq 30$



For firm 2 to choose $Q_2 = 32$, it must believe that firm 1's average quantity is $\tilde{Q}_1 = 26$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

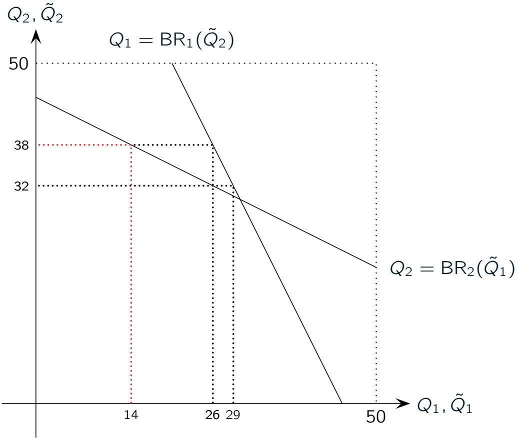
Attempt at rationalizing $Q_i \neq 30$



For firm 1 to choose $Q_1 = 26$, it must believe that firm 2's average quantity is $\tilde{Q}_2 = 38$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

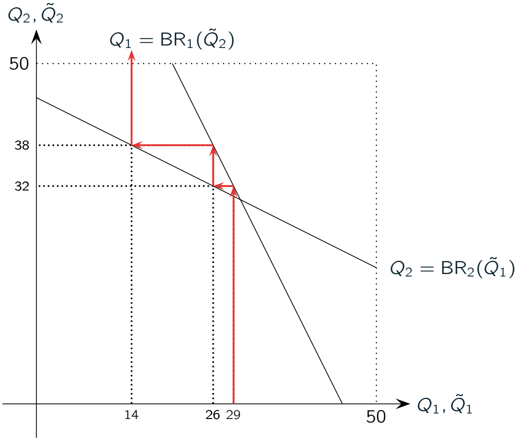
Attempt at rationalizing $Q_i \neq 30$



For firm 2 to choose $Q_2 = 38$, it must believe that firm 1's average quantity is $\tilde{Q}_1 = 14$ which is never rational because firm 1 will always choose $Q_1 > 20$

Example: $P_0 = 100,\ \bar{Q} = 50,\ k = 10$

Attempt at rationalizing $Q_i \neq 30$



Hence firm 1 cannot rationalize choosing $Q_1 = 29$

Bertand competition

- Two firms indexed by $i \in I = \{1, 2\}$ sell similar commodities
- Firms simultaneously choose prices in $P_i \in \mathbb{R}_+$
- The demand for each firm's commodity is determined by the market according to:

$$D_i(P) = D_0 - P_i - \gamma P_{-i}$$

- Firms have constant marginal cost $k \in \mathbb{R}_{++}$ so that their profits are given by:

$$u_i(P) = \Big(P_i - k\Big) D_i(P) = -P_i^2 + (D_0 - \gamma P_{-i}) P_i + k(\gamma P_{-i} - D_0)$$

Proposition

The unique rationalizable prices are $P_i = \dfrac{D_0}{2 + \gamma}$

[0]

Location game

- Henry and George are ice-cream vendors, they sell identical products at identical prices
- On a sunny day the must choose a location for their vending carts along the beach
- Suppose that the beach line is divided into 7 uniformly spaced regions
- On each region there are 10 costumers that will buy ice-cream for the closest vendor (splitting evenly if the vendors are at equal distance)
- Henry and George choose their location simultaneously and their payoff is $1 for each costumer that buys from them

Payoff matrix

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

Iterated dominance

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

Can Henry rationalize choosing 1 or 7?
NO, because 1 is strictly dominated by 2 and 7 is strictly dominated by 6.

Iterated dominance

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

Can Henry rationalize choosing 1 or 7?

NO, because 1 is strictly dominated by 2 and 7 is strictly dominated by 6.

Iterated dominance

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

Can George rationalize choosing 2 or 6?
NO, because *knowing that Henry's location will be between 2 and 6*, 2 is strictly dominated by 3 and 6 is dominated by 5

Iterated dominance

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

Can George rationalize choosing 2 or 6?

NO, because *knowing that Henry's location will be between 2 and 6*, 2 is strictly dominated by 3 and 6 is dominated by 5

Rationalizability

	1	2	3	4	5	6	7
1	35, 35	10, 60	15, 55	20, 50	25, 45	30, 40	35, 35
2	60, 10	35, 35	20, 50	25, 45	30, 40	35, 35	40, 30
3	55, 15	50, 20	35, 35	30, 40	35, 35	40, 30	45, 25
4	50, 20	45, 25	40, 30	35, 35	40, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, 40	35, 35	50, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, 60	35, 35	60, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, 60	35, 35

In fact the only rationalizable strategy for either player is locating at the middle of the beach, i.e. choosing 4. This result is very general and is known as *the median voter theorem*

Thanks

This concludes the second part of the course!

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