Take home exam – Answer key

Knowledge, belief and rationality
New Economic School and Higher School of Economics
Summer 2013

Due on 07 · 25 · 2013

Problem 1. Unawareness

"Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones."

– Donald Rumsfeld

For this problem consider the single-agent epistemic model from Dekel et al. (1998). The model consists of a finite set of states of the world Ω, and two correspondences $\mathsf{K}, \mathsf{U} : 2^{\Omega} \rightrightarrows 2^{\Omega}$. K is to be thought of as a knowledge correspondence, i.e. $\mathsf{K}(E)$ is the set of states at which the agent knows the event E. U is to be thought of as an unawareness correspondence, i.e. $\mathsf{U}(E)$ is the set of states at which the agent is unaware of the event E. Being unaware of E means that the agent does not know E, and also does not know that she does not know E. This idea is captured by the following condition:

$$\forall E \subseteq \Omega : \qquad \mathsf{U}(E) \cap \mathsf{K}(E) = \emptyset \quad \wedge \quad \mathsf{U}(E) \cap \mathsf{K}\big(\Omega \backslash \mathsf{K}(E)\big) = \emptyset \tag{U0}$$

It may also be desirable to impose the following conditions:

$$\forall E \subseteq \Omega : \qquad \mathsf{K}\big(\mathsf{U}(E)\big) = \emptyset \tag{U1}$$

$$\forall E \subseteq \Omega : \qquad \mathsf{U}(E) \subseteq \mathsf{U}\big(\mathsf{U}(E)\big) \tag{U2}$$

1.a) Prove that if the model satisfies (U0) and the monotonicity and truth axioms (conditions (K2) and (K3) in the class slides), then it also satisfies (U1).

Proof. Fix some $E \subseteq \Omega$.

$$\mathsf{U}0 \models \quad \mathsf{U}(E) \subseteq \Omega \backslash \mathsf{K}(E) \tag{1}$$

$$1, K2 \models \quad \mathsf{KU}(E) \subseteq \mathsf{K}(\Omega \backslash \mathsf{K}(E)) \tag{2}$$

$$\mathsf{U}0 \models \quad \mathsf{U}(E) \subseteq \Omega \backslash \mathsf{K}(\Omega \backslash \mathsf{K}(E)) \tag{3}$$

$$3, K3 \models \quad \mathsf{KU}(E) \subseteq \mathsf{U}(E) \subseteq \Omega \backslash \mathsf{K}\big(\Omega \backslash \mathsf{K}(E)\big) \tag{4}$$

$$2, 4 \models \quad \mathsf{KU}(E) \subseteq \mathsf{K}\big(\Omega \backslash \mathsf{K}(E)\big) \cap \big(\Omega \backslash \mathsf{K}(\Omega \backslash \mathsf{K}(E))\big) = \emptyset \tag{5}$$

∎

1.b) Prove that if the model satisfies (U0)–(U2) and monotonicity (K2), then *the agent cannot know something and be unaware of something else at the same time*, i.e. $U(E) \cap \mathrm{K}(F) = \emptyset$ for all $E, F \in 2^{\Omega}$

Proof. Fix two sets $E, F \subseteq \Omega$.

$$\begin{aligned}
\mathrm{U2}, \mathrm{U0} &\models \quad \mathrm{U}(E) \subseteq \mathrm{UU}(E) \subseteq \Omega \backslash \mathrm{K}(\Omega \backslash \mathrm{KU}(E)) && (6)\\
\mathrm{U1} &\models \quad \Omega \backslash \mathrm{KU}(E) = \Omega && (7)\\
6, 7 &\models \quad \mathrm{U}(E) \subseteq \Omega \backslash \mathrm{K}(\Omega) && (8)\\
K2 &\models \quad \mathrm{K}(F) \subseteq \mathrm{K}(\Omega) && (9)\\
9 &\models \quad \Omega \backslash \mathrm{K}(F) \supseteq \Omega \backslash \mathrm{K}(\Omega) && (10)\\
8, 10 &\models \quad \mathrm{U}(E) \subseteq \Omega \backslash \mathrm{K}(F) && (11)\\
11 &\models \quad \mathrm{U}(E) \cap \mathrm{K}(F) = \emptyset && (12)
\end{aligned}$$

■

1.c) Prove that if the model satisfies (U0)–(U2) and the completeness axiom (K1), then *the agent cannot be unaware of anything*, i.e. $U(E) = \emptyset$ for all $E \subseteq \Omega$.

Proof. Fix some $E \subseteq \Omega$.

$$\begin{aligned}
K1 &\models \quad \Omega \backslash \mathrm{K}(\Omega) = \emptyset && (1)\\
8, 1 &\models \quad \mathrm{U}(E) \subseteq \emptyset && (2)
\end{aligned}$$

■

For this problem consider a two player finite game in strategic form (I, A, u), with $I = \{1, 2\}$. In class, we introduced a notion of strict dominance for such games. This problem asks you to analyze a notion of *weak dominance*, and a notion of *dominated* actions introduced by Borgers (1993). For each player i and every set $B_{-i} \subseteq A_{-i}$, define the *weak dominance* relation conditional on B_{-i} as follows. a_i weakly dominates a'_i with respect to B_{-i} if and only if:

$$\forall a_{-i} \in B_{-i} \qquad u_i(a_i, a_{-i}) \geq u(a'_i, a_{-i}) \tag{WD1}$$
$$\exists a_{-i} \in B_{-i} \qquad u_i(a_i, a_{-i}) > u(a'_i, a_{-i}) \tag{WD2}$$

We say that a_i is *weakly dominated* with respect to B_{-i} if there exists some a_i which weakly dominates it with respect to B_{-i}. When $B_{-i} = A_{-i}$, we omit the reference to B_{-i} and simply say that a'_i weakly dominates a_i, or that a_i is weakly dominated. A strategy a_i is said to be *dominated* if and only if it is weakly dominated with respect to every $B_{-i} \subseteq A_{-i}$.

2.a) Show that if a player has strictly positive beliefs (assigns strictly positive probability to his opponents playing a_{-i} for every $a_{-i} \in A_{-i}$), then his best responses cannot be weakly dominated.

Proof. Suppose that $a_i \in A_i$ is weakly dominated by $a'_i \in A_i$ and let $p \in \Delta(A_{-i})$ be an arbitrary belief such that $p(a_{-i}) > 0$ for all $a_{-i} \in A_{-i}$. By WD1 and WD2, and the fact that $p(a_{-i}) > 0$ for every $a_{-i} \in A_{-i}$, it follows that:

$$\forall a_{-i} \in B_{-i} \qquad p(a_{-i})u_i(a_i, a_{-i}) \leq p(a_{-i})u(a'_i, a_{-i})$$
$$\exists a_{-i} \in B_{-i} \qquad p(a_{-i})u_i(a_i, a_{-i}) < p(a_{-i})u(a'_i, a_{-i})$$

Therefore, adding up over $a_{-i} \in A_{-i}$:

$$U_i(a_i, p) = \sum_{a_{-i} \in A_{-i}} p(a_{-i})u_i(a_i, a_{-i}) < \sum_{a_{-i} \in A_{-i}} p(a_{-i})u_i(a'_i, a_{-i}) = U_i(a'_i, p)$$

Which means that a_i cannot be a best response to p. ■

2.b) Provide an example of a game with a pure strategy Nash equilibrium conformed by weakly dominated strategies.

	A	B
A	1 , 1	0 , 1
B	1 , 0	1 , 1

The game represented by the previous payoff matrix has two NE (A, A) and (B, B). For both players, strategy A is weakly dominated by strategy B.

2.c) Show that a dominated action can never be a best response to any belief.

Proof. Suppose that $a_i \in A_i$ is a best response to some belief $p \in \Delta(A_{-i})$. Let $B_{-i} = \{a_{-i} \in A_{-i} \mid p(a_{-i}) > 0\}$ be the support of p. From (2.a) it follows that a_i is not weakly dominated with respect to B_{-i}, and hence it is not dominated. ■

2.d) Show by example that the order of elimination of *weakly* dominated strategies matters.

	L	R
T	1 , 1	0 , 0
M	1 , 1	2 , 1
B	0 , 0	2 , 1

B is weakly dominated by M. After eliminating B, R is weakly dominated by L. After eliminating R, there are no more dominated strategies and the resulting game is:

	L
T	1 , 1
M	1 , 1

An alternative elimination procedure is as follows. T is weakly dominated by M. After eliminating T, L is weakly dominated by R. After eliminating R, there are no more dominated strategies and the resulting game is:

	R
M	2 , 1
B	2 , 1

Hence there are two elimination procedures leading to different terminal games.

References

Borgers, T. (1993). Pure strategy dominance. *Econometrica*, 61(2):423–430.

Dekel, E., Lipman, B. L., and Rustichini, A. (1998). Standard state-space models preclude unawareness. *Econometrica*, 66(1):159–173.